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# DENSITY OF STATE IN A COMPLEX RANDOM MATRIX THEORY WITH EXTERNAL SOURCE

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**Abstract**

The density of state for a complex $N \times N$ random matrix coupled to an external deterministic source is considered for a finite N, and a compact expression in an integral representation is obtained.

The random matrix theory, in which the eigenvalues of random matrix are complex, may find some applications. For example, the two-dimensional electron systems under a strong magnetic field [1] or the study of neural network [2] is similar to the random matrix theory.

In a long time ago, Ginibre [3] considered the complex random matrix theory and he obtained a density of state $\rho(z)$ ( $z = x + iy$ ); inside the circle in complex plane, the density of states $\rho(z)$ becomes uniformly flat and is vanishing outside the circle. This is a generalization of Wigner's semi-circle law in the large N limit for the complex case.

Recently, the random matrix theory with an external source has been investigated [4-16]. The external source is a deterministic, and non-random matrix, coupled to a random matrix. It has been discussed for a Hermitian random matrix [4-8] and for a chiral case [9]. Feinberg and Zee has studied the complex random matrix with an external source in the large N limit [10]. The asymmetric random matrix with external source has been also studied [10,11]. In the large N limit, the boundary of the density of state in the complex plane may be obtained by several methods. However, the expression for the density of state, in a finite N, is much harder for the external source problem. It is known that there appear interesting transitions of opening a gap by tuning the external source [5,13]. It may be crucial to obtain an exact expression for the density of state in a finite N for such problems.

In this letter, we study a complex random matrix which couples to an external source matrix. We generalize the previous works for the real eigenvalues [6,9] to this complex eigenvalue case. The density of state $\rho(z)$ for the complex eigenvalues $z$ is given by

$$\rho(z) = \frac{1}{N} < \sum_{i=1}^{N} \mathrm{tr}\delta(x - \mathrm{Re}\lambda_i)\delta(y - \mathrm{Im}\lambda_i) > \tag{1}$$

where $\lambda_i$ is an eigenvalue of a complex matrix $M$, which couples to the external source matrix $A$ through the following probability distribution for the present case,

$$P_A(M) = \frac{1}{Z_A} e^{-N\mathrm{tr}M^\dagger M + N\mathrm{tr}(M^\dagger A + A^\dagger M)} \tag{2}$$

It has obtained by Ginibre [3] for a finite N, and $A = 0$ as

$$\rho(z) = \frac{1}{\pi} \sum_{n=0}^{N-1} \frac{N^n |z|^{2n}}{n!} e^{-N|z|^2} \tag{3}$$

where we write the result in which the radius of the disk is unity in the large N limit as a normalization. To evaluate the density of state (1), it is useful to consider a chiral Hamiltonian,

$$H = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix} \tag{4}$$

where $M$ is a complex matrix. We denote the density of state of this chiral Hamiltonian by $\rho_{ch}(\lambda)$,

$$\rho_{ch}(\lambda) = \frac{1}{2N} < \mathrm{tr}\delta(\lambda - H) > \tag{5}$$

where the probability distribution is $P(H) = \frac{1}{Z}\exp[-N\mathrm{tr}M^\dagger M]$. Note that the eigenvalues of $H$ are always real and appear in pairs of positive and negative values. Due to this chirality of the eigenvalues, the density of state $\rho_{ch}(\lambda)$ is equal to

$$\rho_{ch}(\lambda) = |\lambda|\tilde{\rho}(\lambda^2) \tag{6}$$

where

$$\tilde{\rho}(r) = \frac{1}{N} < \mathrm{tr}\delta(r - M^\dagger M) > \tag{7}$$

in which the average distribution probability $P(M)$ is same as $P_A(M)$ in (2).

As noticed by Feinberg and Zee [10], the density of state $\rho(z)$ in (1) is obtained from the expression of the density of state $\rho_{ch}(\lambda)$ by the shift of $A$ in (2) as $A \to A - zI$. Using the well-known expressions for the complex delta-function $\delta(z) = \delta(x)\delta(y)$, z = x + i y,

$$\delta(z - z_0) = \frac{1}{\pi}\frac{\partial}{\partial z^*}(\frac{1}{z - z_0}) \tag{8}$$

$$\pi\delta(z) = \partial_z\partial_{z^*}\log(zz^*), \tag{9}$$

we have

$$\rho(z) = \frac{1}{\pi}\partial_z\partial_{z^*} < \frac{1}{N}\mathrm{trlog}(z - M)(z^* - M^\dagger) > \tag{10}$$

Using a dispersion relation between Green function and the density of state $\rho_{ch}(\lambda)$, we get

$$\begin{aligned} \rho(z) &= -\frac{2i}{\pi}\int_0^\infty ds\partial_z\partial_{z^*}(\int_{-\infty}^\infty \frac{\rho_{ch}(\lambda)}{is - \lambda}d\lambda) \\ &= \frac{4}{\pi}\partial_z\partial_{z^*}\int_0^\infty ds\int_0^\infty d\lambda\frac{\lambda s}{\lambda^2 + s^2}\tilde{\rho}(\lambda^2) \end{aligned} \tag{11}$$

in which the external source $A$ is shifted as $A = diag(|a_1 - z|e^{i\theta_1}, ..., |a_N - z|e^{i\theta_N})$. In the large N limit, this $\tilde{\rho}(\lambda^2)$ was obtained by a diagrammatic analysis, and the density of state $\rho(z)$ was obtained by this procedure [10]. We consider here the finite N case, not in the large N limit, by calculating the chiral $\rho_{ch}(\lambda)$ with an external source through Itzykson-Zuber integral [17].

A complex matrix $M$ is decomposed as

$$M = UXV \tag{12}$$

where $U$ and $V$ are unitary matrices and $X$ ia a diagonal matrix. Since the number of real variables is $2N^2$ for $M$, $N^2$ for $U$,$V$ and $2N$ for X, we have $2N$ conditions on $U$,$V$ and $X$. It is possible to put the condition that the diagonal element of $V$ is real, and $X = diag(x_1, ..., x_N)$, $x_i$ is real, $x_i \geq 0$. Note that $x_i$ is not an eigenvalue of $M$, but $x_i^2$ is an eigenvalue of $M^\dagger M$. $x_i$ is called a singular value of $M$.

Itzykson-Zuber integral for this case is known [18-20],

$$\int dUdV e^{\mathrm{Re}(\mathrm{tr}UXVY)} = \frac{(2\pi)^{N^2}}{N!}\frac{\det[I_0(x_i y_j)]}{\Delta(x^2)\Delta(y^2)} \tag{13}$$

where $Y = diag(y_1, ..., y_N)$ and $\Delta(x^2) = \prod_{i<j}(x_i^2 - x_j^2)$, which is a Van der Monde determinant. This Itzykson-Zuber integral is obtained by applying a Laplacian of $M$ to (13). This Laplacian reduces to the diagonal one, and (13) is a zonal spherical function.

The external source $A = diag(|a_1 - z|e^{i\theta_1}, ..., |a_N - z|e^{i\theta_N})$ is decomposed as $A = Y\tilde{U}$, $\tilde{U} = diag(e^{i\theta_1}, ..., e^{i\theta_N})$ and $Y = diag(|a_1 - z|, ..., |a_N - z|)$. This phase unitary matrix $\tilde{U}$ can be absorbed in $U$. Thus, we have a diagonal matrix element $y_i = |a_i - z|$ in $Y$.

Using the contour representation method by Kazakov [21], and taking the same procedure by Brézin, Hikami and Zee (BHZ)[9], we evaluate the evolution operator $U_A(t)$, which is a Fourier transform of the density of state $\tilde{\rho}(\lambda)$ in (6), by noting that $x_i^2$ is an eigenvalue of $M^\dagger M$,

$$\tilde{\rho}(\lambda) = \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-it\lambda}U_A(t) \tag{14}$$

$$\begin{aligned} U_A(t) &= \frac{1}{N} < \mathrm{tr}e^{itM^\dagger M} > \\ &= \frac{1}{NZ_A}\sum_{\alpha}^{N}\int_0^\infty dx \prod_{i=1}^{N} x_i \frac{\Delta(x^2)}{\Delta(y^2)}\det[I_0(2Nx_i y_j)]e^{-N\sum x_i^2 + itx_\alpha} \end{aligned} \tag{15}$$

The coefficient of (14) is not important since we normalize $U_A(t)$ as $U_A(0) = 1$. This expression is similar to the previous results [6,8] except that we have a modified Bessel function $I_0(2Nx_iy_j)$ instead of $e^{Nx_iy_j}$ as an element of the determinant. The modified Bessel function has an integral representation as

$$I_0(2\sqrt{a}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{e^{i\theta}+ae^{-i\theta}} \tag{16}$$

Keeping the notation $y_i^2 = |a_i - z|^2$, we find the integral representation for $U_A(t)$ as

$$\begin{aligned} U_A(t) &= \frac{1}{N}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\int_0^\infty dq \oint \frac{du}{2\pi i}\frac{1}{(1-\frac{it}{N}-u)(f-ue^{i\theta})}e^{-q+i\theta+(1-u)e^{i\theta}} \\ &\times \prod_{i=1}^{N}\left(\frac{f-Ny_i^2}{ue^{i\theta}-Ny_i^2}\right) \end{aligned} \tag{17}$$

where $f = (\frac{it+Nu}{N-it-Nu})q$. The contour integral over $u$ is reduced to evaluation of the residue at the pole $u = Ny_i^2 e^{-i\theta}$. The integration over $q$ is intoduced for the absorption of a combinatorial factor $k!$, which appears in the $x_i$ integral in (14).

One can easily find that when there is no external source $y_i = 0$, the expression $U_{A=0}(t)$ in (17) reduces to the result of BHZ [9]. The density of state $\tilde{\rho}(\lambda)$ is given by the shift of $t \to N(t+iu)$,

$$\tilde{\rho}(\lambda) = \int_{-\infty}^{\infty}\frac{dt}{2\pi}\oint\frac{du}{2\pi i}\oint\frac{dv}{2\pi i}\int_0^\infty dq\frac{e^{-q-iNt\lambda+Nu\lambda-v(1-\frac{1}{u})}}{u(1-it)(f-v)}\prod_i\left(\frac{f-Ny_i^2}{v-Ny_i^2}\right) \tag{18}$$

where $f = itq/(1-it)$, and we have made a change of variable $e^{i\theta} = v/u$. Now we consider the density of state $\rho(z)$ for the complex matrix through (11). We replace a factor as

$$\frac{1}{s^2+\lambda^2} = \int_0^\infty e^{-\alpha(s^2+\lambda^2)}d\alpha \tag{19}$$

Inserting (18) into (11), and doing the integration over $s, \lambda$ and $t$, we obtain by the change of variables, $u \to u+\alpha/N$, $q \to (1-u)q/u$ and $\alpha \to N\alpha u$ and $\beta = \alpha/(\alpha+1)$,

$$\rho(z) = -\frac{1}{\pi N}\partial_z\partial_{z^*}\left[\int_0^1 d\beta\oint\frac{du}{2\pi i}\oint\frac{dv}{2\pi i}\int_0^\infty dq\frac{e^{-\frac{1-u}{u}q-v+\frac{v}{u}(1-\beta)}}{\beta u^2(q-v)}\prod_{i=1}^{N}\left(\frac{q-Ny_i^2}{v-Ny_i^2}\right)\right] \tag{20}$$

where the contour of the integration of $v$ is around $Ny_i^2$ and the contour of $u$ is around $u = 1$, which appears as a pole after the integration of $q$.

If $f(q)$ is a polynomial of $q$, we are able to prove that

$$\oint \frac{du}{2\pi i} \int_0^\infty dq \frac{1}{u^2} e^{\frac{v(1-\beta)}{u}} e^{-\frac{(1-u)q}{u}} f(q) = -e^{v(1-\beta)} f(v(1-\beta)) \tag{21}$$

Thus, the integrations over $q$ and $u$ can be done, and we finally obtain by the shift $v \to Nv$,

$$\rho(z) = \frac{1}{\pi N} \partial_z \partial_{z^*} [\int_0^1 d\beta \oint \frac{dv}{2\pi i} \frac{e^{-\beta N v}}{\beta^2 v} \prod_{i=1}^N (1 - \frac{\beta v}{v - y_i^2})] \tag{22}$$

where contours are taken around all $y_i^2$.

It is easy to write down the explicit form for the small N. We have

$$\begin{aligned}
\rho(z) &= \frac{1}{\pi} e^{-|a_1 - z|^2} \qquad (N = 1) \\
&= \frac{1}{\pi} [e^{-2|a_1 - z|^2} + e^{-2|a_2 - z|^2} - \frac{1}{4} \partial_z \partial_{z^*} (\frac{e^{-2|a_1 - z|^2} - e^{-2|a_2 - z|^2}}{|a_1 - z|^2 - |a_2 - z|^2})] \quad (N = 2) \\
&= \frac{1}{\pi} [\sum_{i=1}^3 e^{-3y_i^2} - \frac{1}{3} \partial_z \partial_{z^*} \oint \frac{du}{2\pi i} \frac{e^{-u}(1 + 3\sum_{i=1}^3 y_i^2 - 2u)}{\prod_{i=1}^3 (u - 3y_i^2)}] \quad (N = 3)
\end{aligned} \tag{23}$$

where $y_i^2 = (a_i - z)(a_i^* - z^*)$. It is also easy to see, when we put $a_i = 0$, we obtain $y^2 = z^* z$, and by the differentiation for $z$ and $z^*$, (22) becomes

$$\rho_N(z) = \frac{1}{\pi} \int_0^1 d\beta \oint \frac{dv}{2\pi i} (Ny^2 v - \frac{1}{\beta})(1 - \frac{\beta v}{v - 1})^N e^{-\beta N v y^2} \tag{24}$$

If we write $Ny^2$ by $s^2$, and put $I_N(s) = \rho_N(z) - \rho_{N-1}(z)$, we find $I_N(s) = \frac{s^{2(N-1)}}{(N-1)!} e^{-s^2}/\pi$. This agrees with Ginibre's result (3). It is immediate to obtain the boundary of the density of state from (22) by the saddle point equation. Taking the derivative of the exponent in the large N limit by $\beta$, and putting $\beta = 1$, which is an end point of the integral, we have as a boundary curve $x^2 + y^2 = 1$ for Ginibre case, and $x^4 + y^4 + 2x^2y^2 - 3x^2 + y^2 = 0$ for the case when the external source eigenvalues are $a_i = \pm 1$, $N/2$ times degenerated.

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